Crux Mathematicorum **8** (Canada)

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Another Unlikely Concurrence

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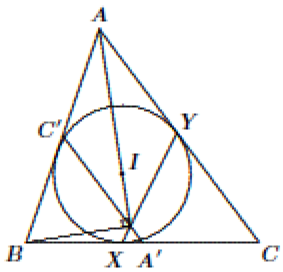
In section 3.4 of [4], under the heading "An Unlikely Concurrence", Ross Honsberger discusses the following theorem:

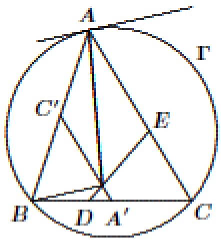
If I is the incentre of $\triangle ABC$, and X and Y the points of contact of the incircle with BC and CA, then the lines AI, XY, and the perpendicular from B to AI are concurrent.



Altshiller Court has this result as an exercise in [1] (p. 118, ex. 43). Earlier, Papelier [6] (p. 19) had stated it for a right triangle. Moreover, long before that (in 1859) Lascases [5] showed that this point also lies on the line that joins the mid-point of BC to the mid-point of BA: more precisely, he proved that the line joining the two mid-points contains the foot of the perpendicular from B to AI. (See [3] p. 327, no. 761.)

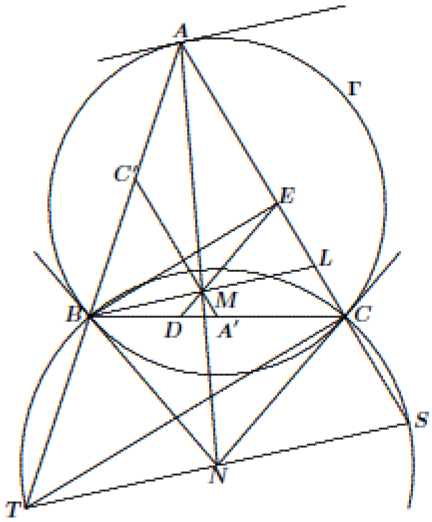
The following theorem gives a similarly unlikely concurrence of four lines—a result which I was unable to locate in the literature.

Theorem. Let ABC be a triangle and Γ its circumcircle. Let D and E be the feet of the altitudes from A to BC and B to CA, and let A' and C' be the respective mid-points of BC and BA. Let s_A be the symmedian through A, and let a'_B be the parallel through B to the tangent to Γ at A. Then the four lines s_A, a'_B, DE, and $A'C'$ are concurrent.



Proof. Let a', b', c' be the tangents to Γ at the vertices A, B, C, respectively; let L, M be the respective intersections of the line a'_B with AC and DE; let N be the intersection of b' and c'; let a'_N be the parallel to a' through N; and let S and T be the respective intersections of a'_N with AC and AB. See the figure on the next page.

[1] *Crux Mathematicorum* (Canada) **8** (2003) 511-513



According to Lemoine, we have $s_A = AN$ (the symmedian through one vertex of a triangle passes through the intersection point of the tangents to the circumcircle at the other two vertices; see [1], p. 248, Theorem 560).

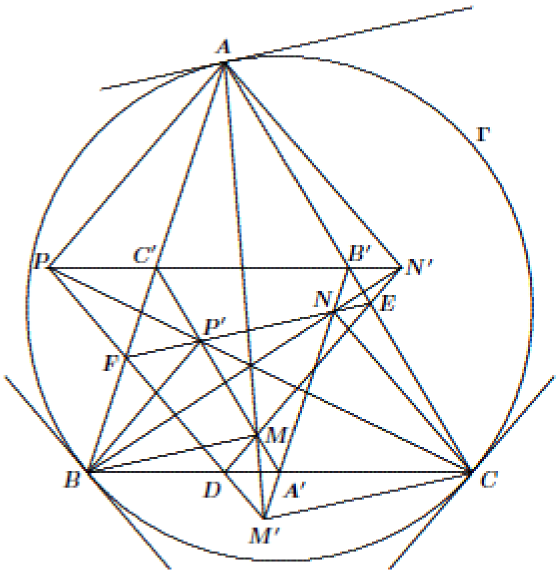
Boutin's Theorem [2] states: If three circles are externally tangent in pairs, the lines joining the contact point of two of the circles to the other two contact points meet the third circle for the second time at the ends of a diameter; furthermore, that diameter is parallel to the line joining the centres of the first two circles. (See [3], p. 553, no. 1256).

Consider now the circle centred at N which passes through B and C, the circle centred at the intersection of a' and b' which passes through A and B, and the circle centred at the intersection of a' and c' which passes through A and C. Boutin's Theorem applied to these circles implies that B, C, S, and T lie on a circle having ST for diameter and N for centre. Consequently, CT is perpendicular to the line ACS. Since $ACS \perp BE$ (by definition), we have $CT \parallel BE$.

A theorem of Carnot (see [3], p. 284) states that the sides of the orthic triangle are parallel to the tangents to the circumcircle of a given triangle at its vertices. Thus, we have $CN \parallel DE$. Also, since $a'_B \parallel a'$ and $a' \parallel a'_N$, we have $a'_B \parallel a'_N$; that is, $BM \parallel TN$. Therefore, the triangles BME and TNC are homothetic. The Theorem of Desargues applied to these triangles (triangles perspective from a line—here the line at infinity—are perspective from a point), implies that A, M, N are collinear.

Since $BL \parallel TS$ and N is the mid-point of TS, M is the mid-point of BL; consequently, M lies on $A'C'$ (since $A'C'$ passes through the mid-points of all cevians through B). Thus, M lies on the four lines as desired.

Of course, we get a second concurrence at M', say, by interchanging the roles of B and C. By cyclically permuting the vertices, we get two more pairs of quadruple concurrence points, N, N' and P, P', indicated in the complete figure below.



References.

[1] Nathan Altshiller Court, *College Geometry*, Barnes and Noble, 1952.

[2] M.A. Boutin, *Journal de mathématiques élémentaires*, (1890), p. 113.

[3] F.G.-M., *Exercices de géométrie*, 4th edition (1907).

[4] Ross Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, New Mathematical Library 37, The Mathematical Association of America, 1995.

[5] A. Lascases, *Nouvelles annales* **18** (1859), p. 171, No. 477.

[6] G. Papelier, *Exercices de géométrie modernes*, Pôles et polaires, 1927.

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