A Purely Synthetic Proof of the Droz-Farny Line Theorem

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Abstract. We present a purely synthetic proof of the theorem on the Droz-Farny line, and a brief biographical note on Arnold Droz-Farny.

1. The Droz-Farny line theorem

In 1899, Arnold Droz-Farny published without proof the following remarkable theorem.

Theorem 1 (Droz-Farny [2]). *If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

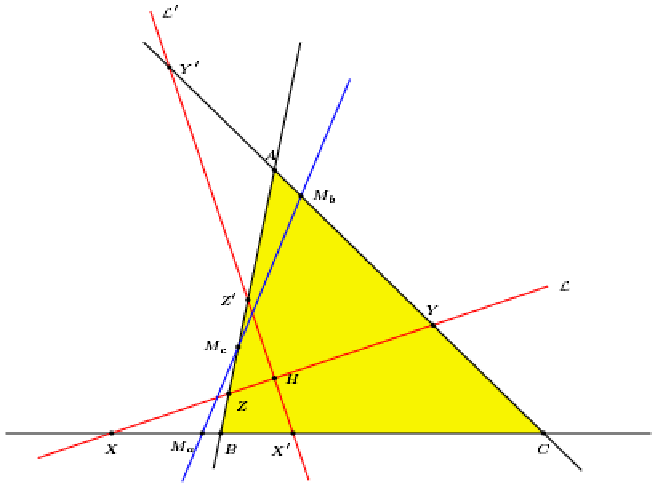

Figure 1.

Figure 1 illustrates the Droz-Farny line theorem. The perpendicular lines $\mathcal{L}$ and $\mathcal{L}'$ through the orthocenter H of triangle ABC intersect the sidelines BC at X, X', CA at Y, Y', and AB at Z, Z' respectively. The midpoints M_a, M_b, M_c of the segments XX', YY', ZZ' are collinear.

It is not known if Droz-Farny himself has given a proof. The Droz-Farny line theorem was presented again without any proof in 1995 by Ross Honsberger [9,

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[1] Ayme J.-L., A purely synthetic proof of the Droz-Farny line theorem, *Forum Geometricorum* (États-Unis) vol. **4**, (2004) 219-224

p.72]. It also appeared in 1986 as Problem II 206 of [16, pp.111,311-313] without references but with an analytic proof. This "remarkable theorem", as it was named by Honsberger, has been the subject of many recent messages in the Hyacinthos group. If Nick Reingold [15] proposes a projective proof of it, he does not yet show that the considered circles intersect on the circumcircle. Darij Grinberg taking up an elegant idea of Floor van Lamoen presents a first trigonometric proof of this "rather difficult theorem" [5, 12, 3] which is based on the pivot theorem and applied on degenerated triangles. Grinberg also offers a second trigonometric proof, which starts from a generalization of the Droz-Farny's theorem simplifying by the way the one of Nicolaos Dergiades and gives a demonstration based on the law of sines [6]. Milorad Stevanović [17] presents a vector proof. Recently, Grinberg [8] picks up an idea in a newsgroup on the internet and proposes a proof using inversion and a second proof using angle chasing. In this note, we present a purely synthetic proof.

2. Three basic theorems

Theorem 2 (Carnot[1, p.101]). *The segment of an altitude from the orthocenter to the side equals its extension from the side to the circumcircle.*

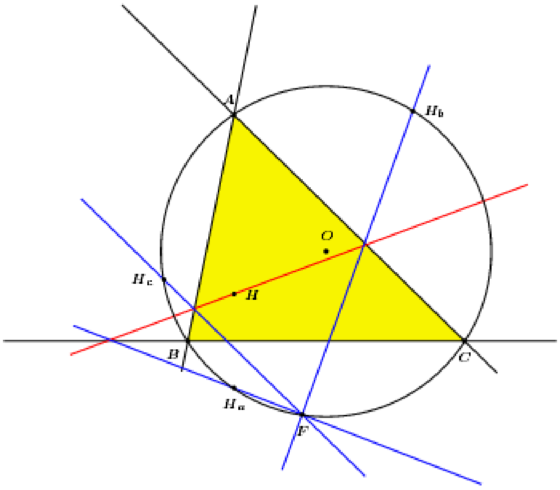

Figure 2.

Theorem 3. *Let $\mathcal{L}$ be a line through the orthocenter of a triangle ABC. The reflections of $\mathcal{L}$ in the sidelines of ABC are concurrent at a point on the circumcircle.*

See [11, p.99] or [10, §333].

Theorem 4 (Miquel's pivot theorem [13]). *If a point is marked on each side of a triangle, and through each vertex of the triangle and the marked points on the adjacent sides a circle is drawn, these three circles meet at a point.*

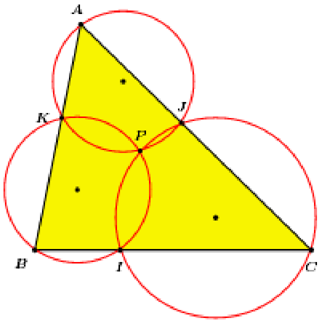

Figure 3.

See also [10, §184, p.131]. This result stays true in the case of tangency of lines or of two circles. Very few geometers contemporary to Miquel had realised that this result was going to become the spring of a large number of theorem.

3. A synthetic proof of Theorem 1

The right triangle case of the Droz-Farny theorem being trivial, we assume triangle ABC not containing a right angle. Let $\mathcal{C}$ be the circumcircle of ABC.

Let $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$) be the circumcircle of triangle HXX' (respectively HYY', HZZ'), and H_a (respectively H_b, H_c) be the symmetric point of H in the line BC (respectively CA, AB). The circles $\mathcal{C}_a$, $\mathcal{C}_b$ and $\mathcal{C}_c$ have centers M_a, M_b and M_c respectively.

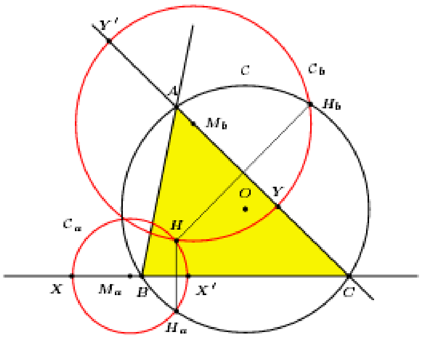

Figure 4.

According to Theorem 2, H_a is on the circle $\mathcal{C}$. XX' being a diameter of the circle $\mathcal{C}_a$, H_a is on the circle. Consequently, H_a is an intersection of $\mathcal{C}$ and $\mathcal{C}_a$, and the perpendicular to BC through H. In the same way, H_b is an intersection of $\mathcal{C}$ and $\mathcal{C}_b$, and the perpendicular to CA through H. See Figure 4.

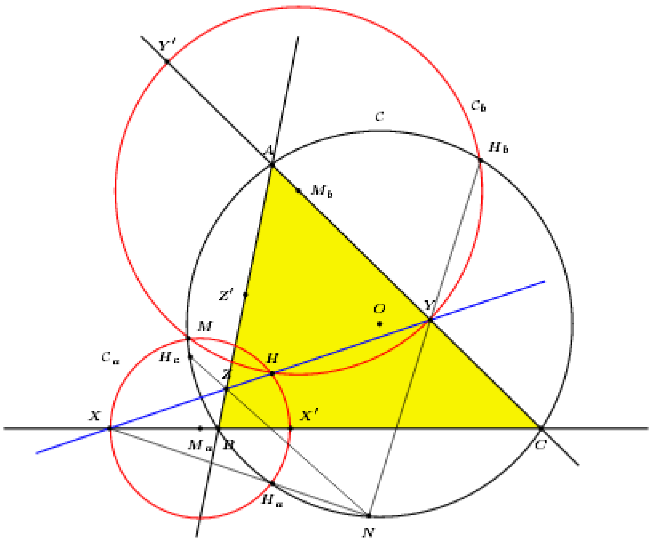

Figure 5.

Consider the point H_c, the symmetric of H in the line AB. According to Theorem 2, H_a is on the circle $\mathcal{C}$. Applying Theorem 3 to the line XYZ through H, we conclude that the lines H_aX, H_bY and H_cZ intersect at a point N on the circle $\mathcal{C}$. See Figure 5.

Applying Theorem 4 to the triangle XNY with the points H_a, H_b and H (on the lines XN, NY and YX respectively), we conclude that the circles $\mathcal{C}$, $\mathcal{C}_a$, and $\mathcal{C}_b$ pass through a common point M.

Mutatis mutandis, we show that the circles $\mathcal{C}$, $\mathcal{C}_b$, and $\mathcal{C}_c$ also pass through the same point M.

The circle $\mathcal{C}_a$, $\mathcal{C}_b$, and $\mathcal{C}_c$, all passing through H and M, are coaxial. Their centers are collinear. This completes the proof of Theorem 1.

4. A biographical note on Arnold Droz-Farny

Arnold Droz, son of Edouard and Louise Droz, was born in La Chaux-de-Fonds (Switzerland) on February 12, 1856. After his studies in the canton of Neufchatel, he went to Munich (Germany) where he attended lectures given by Felix Klein, but he finally preferred geometry. In 1880, he started teaching physics and mathematics in the school of Porrentruy (near Basel) where he stayed until 1908. He is known for having written four books between 1897 and 1909, two of them about geometry. He also published in the *Journal de Mathématiques Élementaires et*

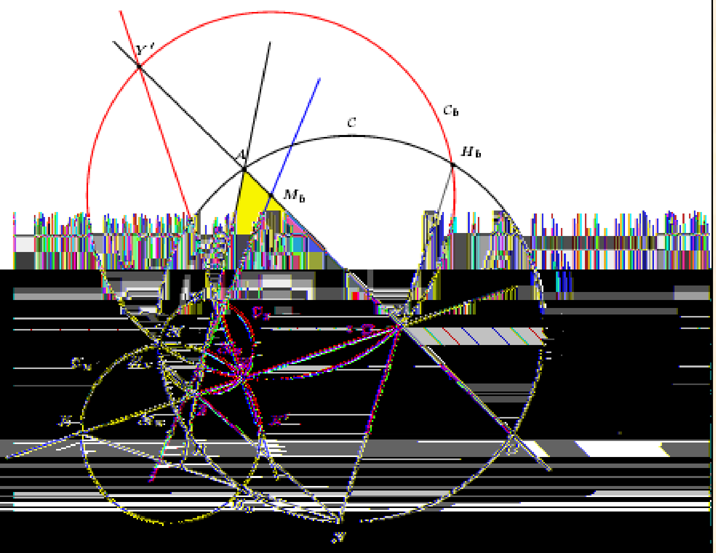

Figure 6.

[illegible] suffered from a long illness. See [4, 14].

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