THE DROZ - FARNY's LINE

THE FIRST PURELY SYNTHETIC PROOF [1]

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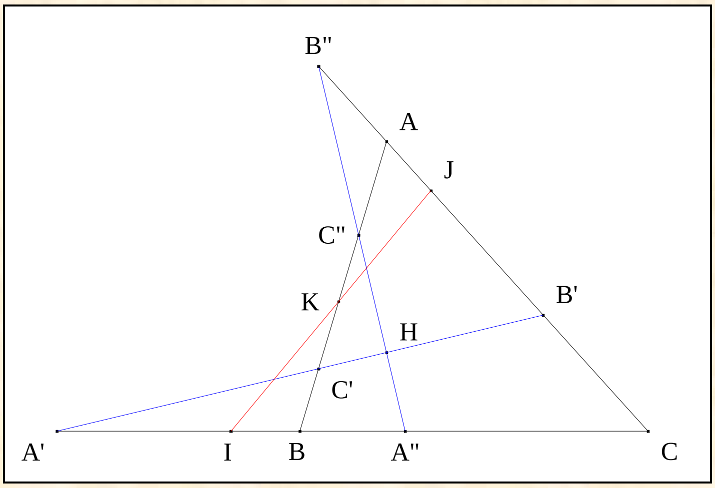

Abstract. We present in this article a purely synthetic proof of the Droz-Farny's line.

Summary

1. Arnold Droz-Farny.[2]

Arnold Droz, son of Edouard and Louise Droz, was born in La Chaux-de-Fonds (Switzerland) on February 12, 1856. After his studies in the canton of Neufchatel, he went to Munich (Germany) where he attended lectures given by Felix Klein, but he finally preferred geometry. In 1880, he started teaching physics and mathematics in the school of Porrentruy (near Basel) where he stayed until 1908.
He is known for having written four books between 1897 and 1909, two of them about geometry. But he also published in the *Journal de Mathématiques Elémentaires et Spéciales* (1894, 1895), and in *L'Intermédiaire des Mathématiciens* and in the *Educational Times* (1899) as well as in *Mathesis* (1901).

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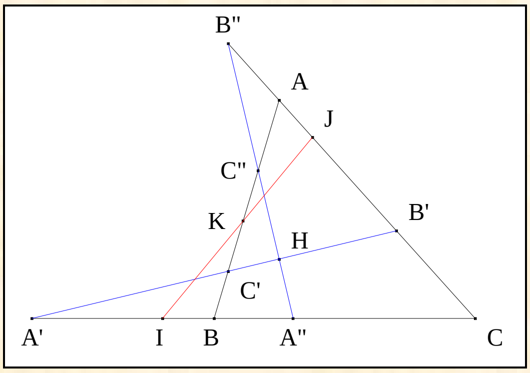
[1] Ayme J.-L., *Forum Geometricorum* (États-unis) **4** (2004) 219-224 ; http://forumgeom.fau.edu/.

[2] J. Gonzalez Cabillon, message to Historia Mathematica, August 18, 2000, available at http://sunsite.utk.edu/math_archives/.http/hypermail/historia/aug00/0064.html.
E. Ortiz, message to Historia Mathematica, August 21, 2000, available at http://sunsite.utk.edu/math_archives/.http/hypermail/historia/aug00/0077.html.

As he was very sociable, he liked to be in contact with other geometers likes the Italian Virginio Retali and the Spanish Juan Jacobo Duran Loriga. In his free time, he liked to climb little mountains and to watch horse races.
He married to Lina Farny who was born also in La Chaux-de-Fonds.
He died in Porrentruy on January 14, 1912 after having suffered from a long illness.

2. The theorem.



If two perpendicular straight lines are drawn through the orthocenter H of a triangle ABC, they intercept a segment on each of the sides i.e. A'A", B'B", C'C", and the midpoints I, J, K of these three segments are collinear [0].
Up to this day, I still don't know if this theorem has been proved or not by Droz-Farny.

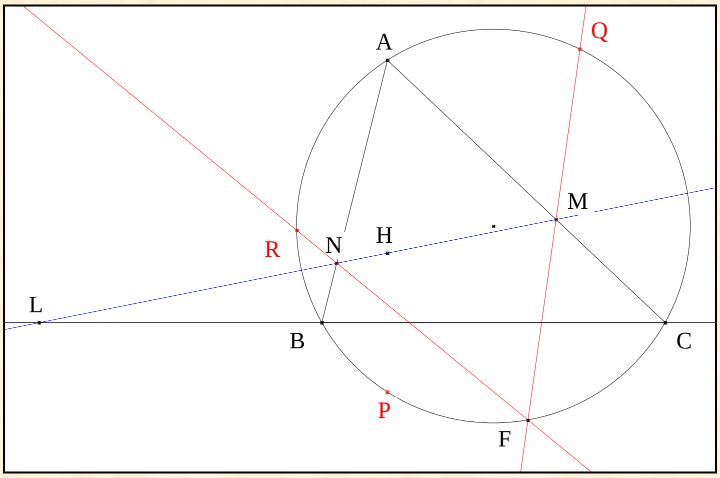
3. The different proofs.

Droz-Farny's line was presented again without any proof in 1995 by Ross Honsberger [1], after having been used by Sharygin [2] in 1986 as an exercise without references but he had succeeded in proving it analytically.
This "remarkable theorem" as it was named by Honsberger in his book has been the subject of many messages inside the Hyacinthos group [3].
If Nick Reingold [4] proposes a projective proof of it, he doesn't yet show that the considered circles intersect on the circumcircle.
Darij Grinberg taking up an elegant idea of Floor van Lamoen presents a first trigonometric proof of this "rather difficult theorem" [5] which is based on the pivot theorem and applied on degenerated triangles. But Grinberg also offers a second trigonometric proof, which starts from a generalisation of the Droz-Farny's theorem simplifying by the way the one of Nicolaos Dergiades and gives a demonstration based on the law of sines [6].
Recently, Milorad Stevanovic presents a vector proof [7].
Recently, Grinberg [7'] picks up an idea in a newsgroup on the internet and proposed a proof using inversion and a second proof using angle chasing. In this note, we present a purely synthetic proof.
Finally, I have discovered the proofs of C. E. Hillyer and Sanjana resp., and also these of Droz-Farny who uses a parabola.

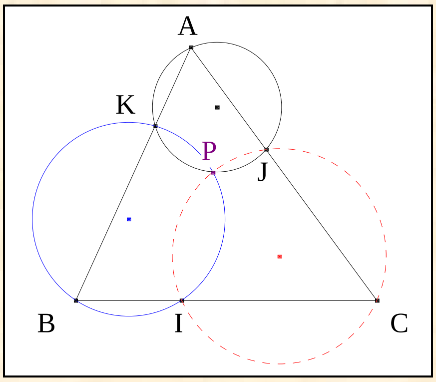
4. The three lemmas.

Lemma 1 (Carnot). The segment of an altitude from the orthocenter to the side equals its extension from the side to the circumcircle [8].

Lemma 2. If a line through the orthocenter H cuts the sides of a triangle ABC at L, M, N, then (PL), (QM), (RN), the reflections of the line with regard to the sides of the triangle, are concurrent at a point F of the circumcircle [9], called the Steiner's antipoint.

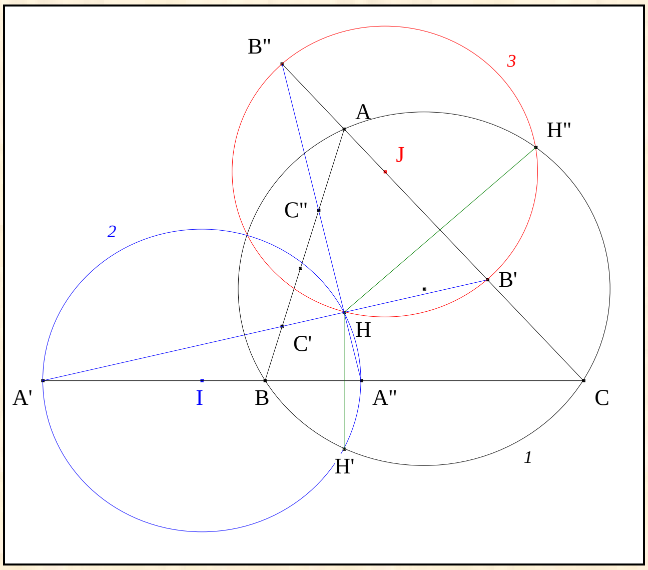


Lemma 3 or the pivot theorem. (Miquel)
If a point is marked on each side of a triangle, and through each vertex of the triangle and the marked points on the adjacent sides a circle is drawn, these three circles meet at a point [10].
This result stays true in the case of tangency of lines or of two circles.
Very few geometers contemporary to Miquel had realised that this result was going to become the spring of a large number of theorem.



5. The synthetic proof.

Situation 1: ABC is not a right triangle

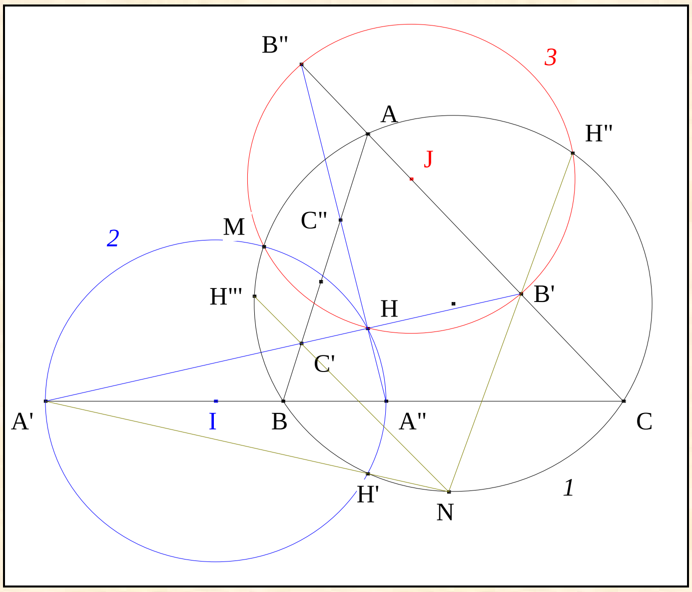


- Let 1 (resp. 2, 3) be the circumcircle of the triangle ABC (resp. HA'A", HB'B")
  and H' (resp. H") be the symmetric point of H in the line BC (resp. CA).

- Let's point out that I (resp. J) is the center of *2* (resp. *3*).

- According to lemma 1, H" is on the circle *1*;
  A'A" being a diameter of the circle *2*, H' is on the circle *2*;
  consequently, H' is the intersection of the circle *1*, the circle *2*, and the perpendicular to BC through H.

- In the same way, we would showed that
  H" is the intersection of the circle *1*, the circle *3*, and the perpendicular to CA through H.

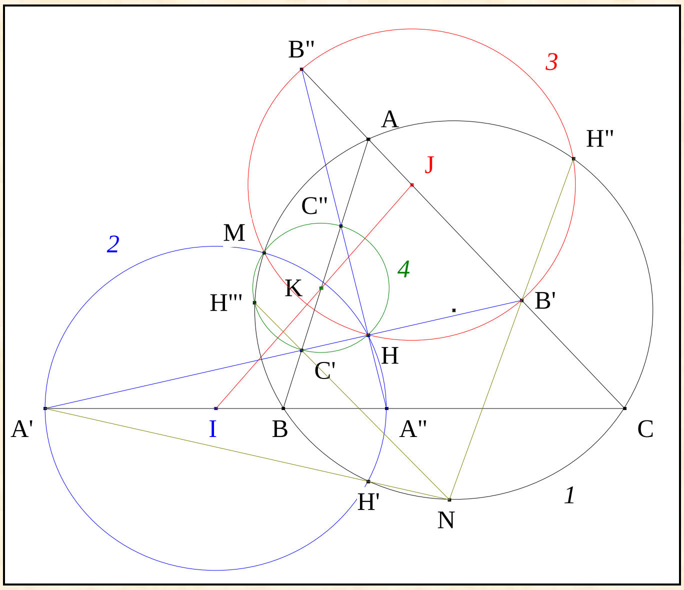


- Let H''' be the symmetric of H relatively to the line AB.

- According to lemma 1, H' is on the circle *1*;
  According to lemma 2 applied to H situated on the line A'B'C',
  the lines H'A', H"B' and H'''C' intersect on the circle *1*.

- Let N be the point of intersection of the lines H'A' and H"B'.

- According to lemma 3 applied to the triangle A'NB' with the points H', H'' and H situated respectively on the lines A'N, NB' and B'A', the circles *1*, *2* et *3* pass through a common point.



- Let M be this point of concurs
  and *4* the circumcircle of the triangle HC'C''.

- Let's point out that K is the center of *4*.

- Mutatis mutandis, we would show that the circles *1*, *3* and *4* pass through the same point M.

- **Conclusion:** the circles *2*, *3* and *4*, all passing through H and M, are coaxial.
  Their centers I, J and K are collinear.

Situation 2: ABC is a right triangle

- This leads to a special case in which situation the Droz-Farny's theorem is trivial.

6. **References** (historical and academic)

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