Sawayama and Thébault's theorem

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Abstract. We present a purely synthetic proof of Thébault's theorem, known earlier to Y. Sawayama.

1. Introduction

In 1938 in a "Problems and Solutions" section of the Monthly [24], the famous French problemist Victor Thébault (1882-1960) proposed a problem about three circles with collinear centers (see Figure 1) to which he added a correct ratio and a relation which finally turned out to be wrong. The date of the first three metric

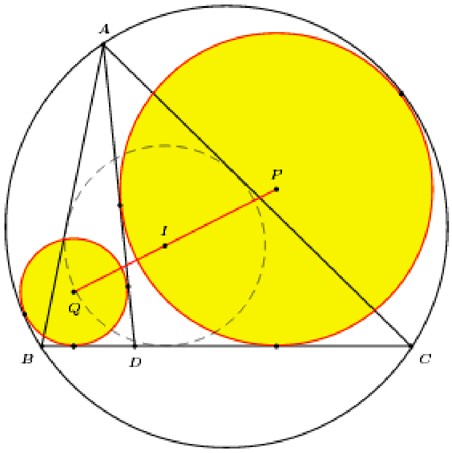

Figure 1

solutions [22] which appeared discreetly in 1973 in the Netherlands was more widely known in 1989 when the Canadian revue *Crux Mathematicorum* [27] published the simplified solution by Veldkamp who was one of the two first authors to prove the theorem in the Netherlands [26, 5, 6]. It was necessary to wait until the end of this same year when the Swiss R. Stark, a teacher of the Kantonsschule of Schaffhausen, published in the Helvetic revue *Elemente der Mathematik* [21] the first synthetic solution of a "more general problem" in which the one of Thébault's appeared as a particular case. This generalization, which gives a special importance to a rectangle known by J. Neuberg [15], citing [4], has been pointed out in 1983 by the editorial comment of the *Monthly* in an outline publication about the supposed

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first metric solution of the English K. B. Taylor [23] which amounted to 24 pages. In 1986, a much shorter proof [25], due to Gerhard Turnwald, appeared. In 2001, R. Shail considered in his analytic approach, a "more complete" problem [19] in which the one of Stark appeared as a particular case. This last generalization was studied again by S. Gueron [11] in a metric and less complete way. In 2003, the *Monthly* published the angular solution by B. J. English, received in 1975 and "lost in the mists of time" [7].

Thanks to *JSTOR*, the present author has discovered in an anciant edition of the *Monthly* [18] that the problem of Shail was proposed in 1905 by an instructor Y. Sawayama of the central military School of Tokyo, and geometrically resolved by himself, mixing the synthetic and metric approach. On this basis, we elaborate a new, purely synthetic proof of Sawayama-Thébault theorem which includes several theorems that can all be synthetically proved. The initial step of our approach refers to the beginning of the Sawayama's proof and the end refers to Stark's proof. Furthermore, our point of view leads easily to the Sawayama-Shail result.

2. A lemma

Lemma 1. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. Let P be the center of the circle C_1 which touches DC, DA at E, F and the circumcircle C_2 of ABC at K. Then the chord of contact EF passes through the incenter I of triangle ABC.*

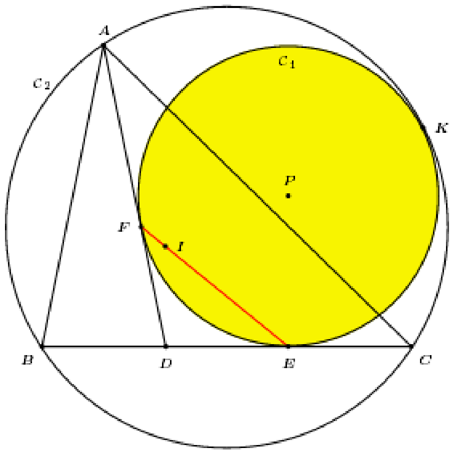

Figure 2

Proof. Let M, N be the points of intersection of KE, KF with C_2, and J the point of intersection of AM and EF (see Figure 3). KE is the internal bisector of $\angle BKC$ [8, Théorème 119]. The point M being the midpoint of the arc BC which does not contain K, AM is the A-internal bisector of ABC and passes through I.

The circles C_1 and C_2 being tangent at K, EF and MN are parallel.

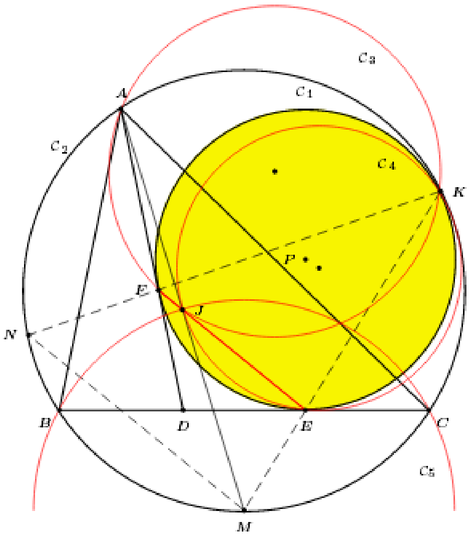

Figure 3

The circle C_2, the basic points A and K, the lines MAJ and NKF, the parallels MN and JF, lead to a converse of Reim's theorem ([8, Théorème 124]). Therefore, the points A, K, F and J are concyclic. This can also be seen directly from the fact that angles FJA and FKA are congruent.

Miquel's pivot theorem [14, 9] applied to the triangle AFJ by considering F on AF, E on FJ, and J on AJ, shows that the circle C_4 passing through E, J and K is tangent to AJ at J. The circle C_5 with center M, passing through B, also passes through I ([2, Livre II, p.46, théorème XXI] and [12, p.185]). This circle being orthogonal to circle C_1 [13, 20] is also orthogonal to circle C_4 ([10, 1]) as KEM is the radical axis of circles C_1 and C_4. [1] Therefore, $MB = MJ$, and $J = I$. Conclusion: the chord of contact EF passes through the incenter I. □

Remark. When D is at B, this is the theorem of Nixon [16].

3. Sawayama-Thébault theorem

Theorem 2. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the center of the incircle of triangle ABC. Let P be the center of the circle which touches DC, DA at E, F, and the circumcircle of ABC, and let Q be the center of a further circle which touches DB, DA in G, H and the circumcircle of ABC. Then P, I and Q are collinear.*

[1]From $\angle BKE = \angle MAC = \angle MBE$, we see that he circumcircle of BKE is tangent to BM at B. So circle C_5 is orthogonal to this circumcircle and consequently also to C_1 as M lies on their radical axis.

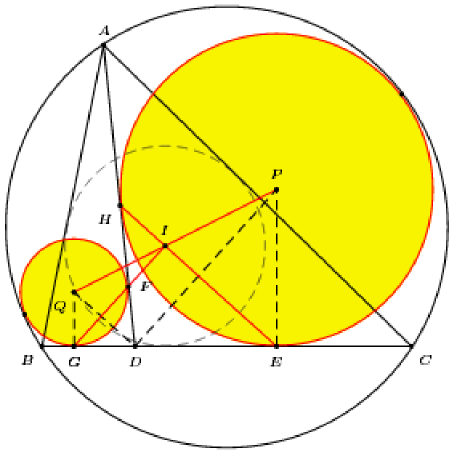

Figure 4

Proof. According to the hypothesis, $QG \perp BC$, $BC \perp PE$; so $QG//PE$. By Lemma 1, GH and EF pass through I. Triangles DHG and QGH being isosceles in D and Q respectively, DQ is
(1) the perpendicular bisector of GH,
(2) the D-internal angle bisector of triangle DHG.

Mutatis mutandis, DP is
(1) the perpendicular bisector of EF,
(2) the D-internal angle bisector of triangle DEF.

As the bisectors of two adjacent and supplementary angles are perpendicular, we have $DQ \perp DP$. Therefore, $GH//DP$ and $DQ//EF$. Conclusion: using the converse of Pappus's theorem ([17, Proposition 139] and [3, p.67]), applied to the hexagon $PEIGQDP$, the points P, I and Q are collinear. □

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