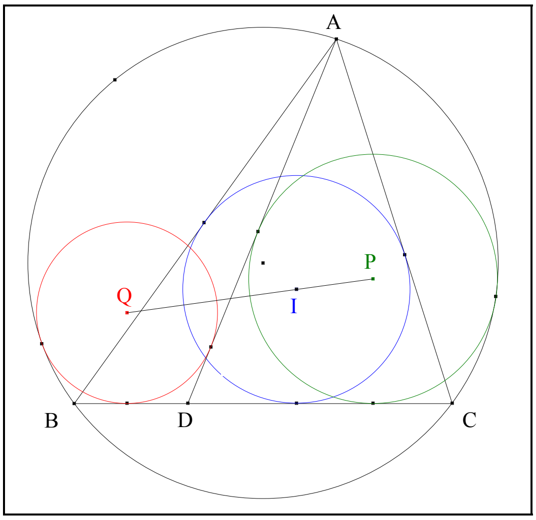
SAWAYAMA or THEBAULT's THEOREM [1]

Jean-Louis Ayme

Abstract. We present a purely synthetic proof of Thébault's theorem, known earlier to Y. Sawayama.

1. Introduction

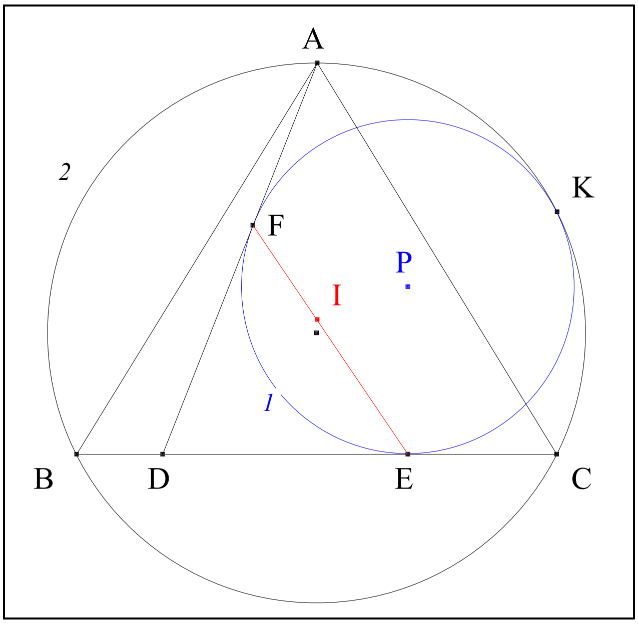
In 1938 in a "Problems and Solutions" section of the *Monthly* [24], the famous French problemist Victor Thébault (1882-1960) proposed a problem about three circles with collinear centers (see Figure 1) to which he added a correct ratio and a relation which finally turned out to be wrong. The date of the first three metric solutions [22] which appeared discreetly in 1973 in the Netherlands was more widely known in 1989 when the



canadian revue *Crux Mathematicorum* [27] pubished the simplified solution by Veldkamp who was one of the two first authors to prouve the theorem in the Netherlands [26, 5, 6]. It was necessary to wait until the end of this same year when the swiss R. Stark, a teacher of the Kantonsschule of Schaffhausen published in the Helvetic revue *Elemente der Mathematik* [21] the first synthetic solution of a "more general problem" in which the one of Thebault appeared as a particular case. This generalization which gives a special importance to a rectangle known by J. Neuberg [15], citing [4], has been pointed out in 1983 by the editorial comment of the *Monthly* in an outline publication about the supposed first metric solution of the english K. B. Taylor [23] which amounted up to 24 pages. In 2001, R. Shail considered in his analytic approach, a "more complete" problem [19] in which the one of Stark appeared like a particular case. This last generalization was studied again by S. Gueron [11] in a metric and less complete way. In 2003, the *Monthly* published the angular solution of B. J. English, received in 1975 and "lost in the mist of time" [7].
Thanks to *JSTOR*, the present author has discovered in a anciant edition of the *Monthly* [18] that the problem of Shail was proposed in 1905 by an instructor Y. Sawayama of the central military School of Tokio, and geometrically resolved by himself, mixing the synthetic and metric approach. On this base, we elaborate a new, purely synthetic proof of Sawayama-Thebault's theorem which includes several theorems that can all be synthetically proved. The initial step of our approach refers to the beginning of the Sawayama's proof and the end refers to Stark's proof. Furthermore, our point of view leads easily to the Sawayama-Shail result.

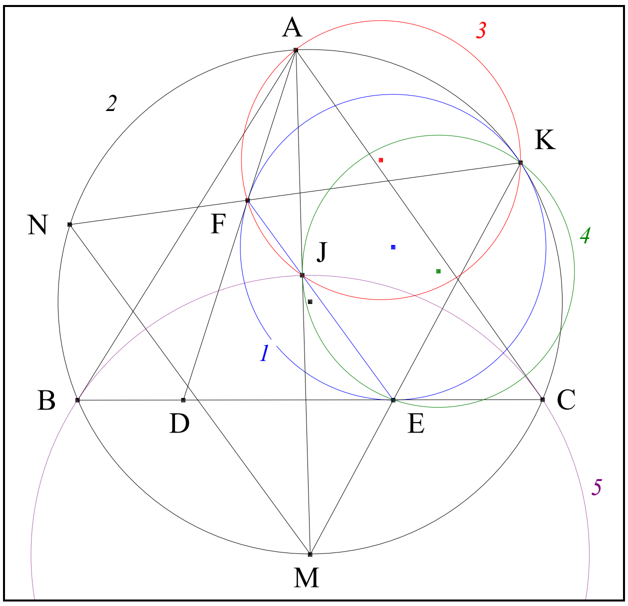
[1] Ayme J. -L., Sawayama and Thébault's theorem, *Forum Geometricorum* vol. 3 (2003) 225-229.

2. A Lemma



Lemma 1. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. Let P be the center of circle 1 which touches DC, DA at E, F and the circumcircle 2 of ABC at K. Then the chord of contact EF passes through the incenter I of triangle ABC.*

Proof. Let M, N be the points of intersection of KE, KF with *2*, and J the point of intersection of AM and EF (see Figure 3). KE is the internal bisector of <BKC [8, Théorème 119]. The point M being the midpoint of the arc BC which does not contain K, AM is the A-internal bisector of ABC and passes through I.
The circles *1* and *2* being tangent at K, EF and MN are parallel.



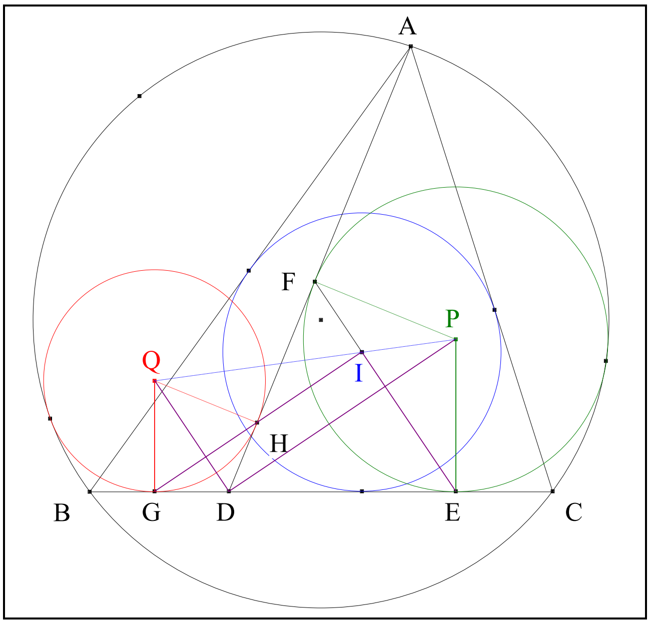
The circle *2*, the basic points A and K, the lines MAJ and NKF, the parallels MN and JF, lead to a converse of Reim's theorem [8, Théorème 124]. Therefore, the points A, K, F and J are concyclic. This can also be seen directly from the fact that angles FJA and FKA are congruent.
Miquel's pivot theorem [14, 9] applied to the triangle AFJ by considering F on AF, E on FJ and J on AJ, shows that the circle *4* passing through E, J and K is tangent to AJ at J. The circle *5* with centre at M, passing through

B, also passes through I ([2, Livre II, p.46, théorème XXI] and [12]). This circle being orthogonal to circle *1* [13, 20][2] , is also orthogonal to circle *4* [10, 1]. Therefore, MB=MJ, and J=I.
Conclusion: the chord of contact EF passes through the incenter I.

Remark. When D is at B, this is the theorem of Nixon [16].

3. Sawayama-Thébault theorem

Theorem 2. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the center of the incircle of ΔABC. Let P be the center of the circle which touches DC, DA at E, F and the circumcircle of ABC, and let Q be the center of a further circle which touches DB, DA in G, H and the circumcircle of ABC. Then P, I and Q are collinear.*



Proof. According to the hypothesis, QG ⊥ BC, BC ⊥ PE ; so (QG) // (PE). By Lemma 1, GH and EF pass through I. Triangles DHG and QGH being isoceles in D and Q respectively, DG is

(1) the perpendicular bisector of GH,
(2) the D-internal angle bisector of triangle DHG.

Mutatis mutandis, DP is

(1) the perpendicular bisector of EF,
(2) the D-internal angle bisector of triangle DEF.

As the bisectors of two adjacent and supplementary angles are perpendicular, we have DQ ⊥ DP.
Therefore, GH // DP and DQ // EF. Conclusion: using the converse of Pappus's theorem ([17, Proposition 139] and [3]), applied to the hexagon PEIGQDP, the points P, I and Q are collinear.

References

[1] N. Altshiller-Court, *College Geometry*, Barnes & Noble, 205.
[2] E. Catalan, Livre II, théorème XXI, *Théorèmes et problèmes de Géométrie élémentaires* (1879) 46.
[3] H. S. M. Coxeter, S. L. Greitzer, *Geometry Revisited*, MAA (1967) 67.
[4] *Archiv der Mathematik und Physik* (1842) 328.
[5] B. C. Dijkstra-Kluyver, Twee oude vraagstukken in één klap opgelost, *Nieuw Tijdschrift voor Wiskunde* 61 (1973-74) 134-135.
[6] B. C. Dijkstra-Kluyver and H. Streefkerk, Nogmaals het vraagstuk van Thébault, *Nieuw Tijdschrift voor Wiskunde* 61 (1973-74) 172-173.

[2] From <BKE = <MAC = <MBE, we see that the circumcircle of BKE is tangent to BM at B. So circle *5* is orthogonal to this circumcircle and consequently also to *1* as M lies on their radical axis.

[7] B. J. English, Solution of Problem 3887, *Amer. Math. Monthly* 110 (2003) 156-158.
[8] F. G.-M., *Exercices de Géométrie*, sixième édition (1920), J. Gabay reprint, 283. Théorème 124.
[9] H. G. Forder, *Geometry*, Hutchinson, 1960, 17.
[10] L. Gaultier (de Tours), Les contacts des cercles, *Journal de l'École Polytechnique*, Cahier 16 (1813) 124-214.
[11] S. Gueron, Two Applications of the Generalized Ptolemy Theorem, *Amer. Math. Monthly* 109 (2002) 362-370.
[12] R. A. Johnson Roger, *Advanced Euclidean Geometry*, Dover, (1965) 185.
[13] *Leybourn's Mathematical repository* (Nouvelle série) 6 tome I, 209.
[14] A. Miquel, Théorèmes de Géométrie, *Journal de mathématiques pures et appliquées* de Liouville 3 (1838) 485-487.
[15] J. Neuberg, *Nouvelle correspondance mathématique* 1 (1874) 96.
[16] R. C. J. Nixon, Question 10693, *Reprints of Educational Times*, London (1863-1918) 55 (1891) 107.
[17] Pappus, *La collection mathématique*, 2 volumes, French translation by Paul Ver Eecke, Paris, Desclée de Brouwer (1933).
[18] Y. Sawayama, A new geometrical proposition, *Amer. Math. Monthly*, 12 (1905) 222-224.
[19] R. Shail., A proof of Thébault's Theorem, *Amer. Math. Monthly*, 108 (2001) 319-325.
[20] S. Shirali, On the generalized Ptolemy theorem, théorème 2, *Crux Math.*, 22 (1996) 48-53.
[21] R. Stark, Eine weitere Lösung der Thébault'schen Aufgabe, *Elem. Math.*, 44 (1989) 130-133.
[22] H.Streefkerk, Waarom eenvougig als het ook ingewikkeld kan?, *Nieuw Tijdschrift voor Wiskunde* **60** (1973-73) 240-253.
[23] K. B. Taylor, Solution of Problem 3887, *Amer. Math. Monthly*, 90 (1983) 482-487.
[24] V. Thébault, Problem 3887, Three circles with collinear centers, *Amer. Math. Monthly*, 45 (1938) 482-483.
[25] G. Turnwald, Über eine Vermutung von Thébault, *Elem. Math.*, 41 (1986) 11-13.
[26] G. R. Veldkamp, Een vraagstuk van Thébault uit 1938, *Nieuw Tijdschrift voor Wiskunde*, 61 (1973-74) 86-89.
[27] G. R. Veldkamp, Solution 1260, *Crux Math.*, 15 (1989) 51-53.